

Coherent state quantization and phase operator

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Abstract

By using a coherent state quantization *à la* Klauder-Berezin, phase operators are constructed in finite Hilbert subspaces of the Hilbert space of Fourier series. The study of infinite dimensional limits of mean values of some observables leads towards a simpler convergence to the canonical commutation relations.

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1 Introduction

Since the first attempt by Dirac in 1927 [1] various definitions of phase operator have been proposed with more or less satisfying success in terms of consistency [3,5,6,7,10]. A natural requirement is that phase operator and number operators form a conjugate Heisenberg pair obeying the canonical commutation relation

$$[\hat{N}, \hat{\theta}] = iI_d, \quad (1)$$

in exact correspondence with the Poisson bracket for the classical action angle variables.

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To obtain this quantum-mechanical analog, the polar decomposition of raising and lowering operators

$$\hat{a} = \exp(i\hat{\theta})\hat{N}^{1/2}, \quad \hat{a}^\dagger = \hat{N}^{1/2}\exp(-i\hat{\theta}), \quad (2)$$

was originally proposed by Dirac, with the corresponding uncertainty relation

$$\Delta\hat{\theta}\Delta\hat{N} \geq \frac{1}{2}. \quad (3)$$

But the relation between operators (1) is misleading. The construction of a unitary operator is a delicate procedure and there are three main problems in it. First we have that for a well-defined number state the uncertainty of the phase would be greater than 2π . This inconvenience, also present in the quantization of the pair angular momentum-angle, adds to the well-known contradiction lying in the matrix elements of the commutator

$$-i\delta_{nn'} = \langle n'|[\hat{N}, \hat{\theta}]|n\rangle = (n - n')\langle n'|\hat{\theta}|n\rangle. \quad (4)$$

In the angular momentum case, this contradiction is avoided to a certain extent by introducing a proper periodical variable $\hat{\Phi}(\phi)$ [4]. If $\hat{\Phi}$ is just a sawtooth function, the discontinuities give a commutation relation

$$[\hat{L}_z, \hat{\Phi}] = -i\{1 - 2\pi \sum_{n=-\infty}^{\infty} \delta(\phi - (2n+1)\pi)\}. \quad (5)$$

The singularities in (5) can be excluded, as proposed by Louisel [2], taking sine and cosine functions of ϕ to recover a valid uncertainty relation. But the problem reveals to be harder in number-phase case because, as showed by Susskind and Glogower (1964)[3], the decomposition (2) itself leads to the definition of non unitary operators:

$$\exp(-i\hat{\theta}) = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \{+|\psi\rangle\langle 0|\}, \text{ and h.c.}, \quad (6)$$

and this non-unitarity explains the inconsistency revealed in (4). To overcome this handicap, a different polar decomposition was suggested in [3]

$$\hat{a} = (\hat{N} + 1)^{\frac{1}{2}}\hat{E}_-, \quad \hat{a}^\dagger = (\hat{N} + 1)^{\frac{1}{2}}\hat{E}_+, \quad (7)$$

where the operators E_\pm are still non unitary because of their action on the extreme state of the semi-bounded number basis [4]. Nevertheless the addition of the restriction

$$\hat{E}_-|0\rangle = 0, \quad (8)$$

permits to define hermitian operators

$$\begin{aligned}\hat{C} &= \frac{1}{2}(\hat{E}_- + \hat{E}_+) = \hat{C}^\dagger \\ \hat{S} &= \frac{1}{2\pi}(\hat{E}_- - \hat{E}_+) = \hat{S}^\dagger.\end{aligned}\tag{9}$$

These operators are named “cosine” and “sine” because they reproduce the same algebraic structure as the projections of the classical state in the phase space of the oscillator problem.

Searching for a hermitian phase operator $\hat{\theta}$ which would avoid constraints like (8) and fit (1) in the classical limit, Popov and Yarunin [6] and later Pegg and Barnett [7] used an orthonormal set of eigenstates of $\hat{\theta}$ defined on the number state basis as

$$|\theta_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\theta_m} |n\rangle.\tag{10}$$

where, for a given finite N , these authors selected the following equidistant subset of the angle parameter

$$\theta_m = \theta_0 + \frac{2\pi m}{N}, \quad m = 0, 1, \dots, N-1,\tag{11}$$

with θ_0 as a reference phase. Orthonormality stems from the well-known properties of the roots of the unity as happens with the base of discrete Fourier transform

$$\sum_{n=0}^{N-1} e^{in(\theta_m - \theta_{m'})} = \sum_{n=0}^{N-1} e^{i2\pi(m-m')\frac{n}{N}} = N\delta_{mm'}.\tag{12}$$

The phase operator on \mathbb{C}^N is simply constructed through the spectral resolution

$$\hat{\theta} \equiv \sum_{m=0}^{N-1} \theta_m |\theta_m\rangle \langle \theta_m|.\tag{13}$$

This construction, which amounts to an adequate change of orthonormal basis in \mathbb{C}^N , gives for the ground number state $|0\rangle$ a random phase which avoids some of the drawbacks in previous developments. Note that taking the limit $N \rightarrow \infty$ is questionable within a Hilbertian framework, this process must be understood in terms of mean values restricted to some suitable subspace and the limit has to be taken afterwards. In [7] the pertinence of the states (10) is proved by the expected value of the commutator with the number operator. The problem appears when the limit is taken since it leads to an approximate result.

More recently an interesting approach to the construction of a phase operator has been done by Busch, Lahti and their collaborators within the frame of measurement theory [8][9][10]. Phase observables are constructed here using the sum over an infinite number basis from their original definition.

Here we propose a construction based on a coherent state quantization scheme and not on the arbitrary assumption of a discrete phase nor on an infinite

dimension Hilbert space. This will produce a suitable commutation relation at the infinite dimensional limit, still at the level of mean values.

2 The approach via coherent state quantization

As was suggested in [7] the commutation relation will approximate better the canonical one (1) if one enlarges enough the Hilbert space of states. We show here that there is no need to discretize the angle variable as in [7] to recover a suitable commutation relation. We adopt instead the Hilbert space $L^2(S^1)$ of square integrable functions on the circle as the natural framework for defining an appropriate phase operator in a finite dimensional subspace. Let us first give an outline of the method already exposed in [11,12,13,14,15].

Let $X = \{x \mid x \in X\}$ be a set equipped with a measure $\mu(dx)$ and $L^2(X, \mu)$ the Hilbert space of square integrable functions $f(x)$ on X :

$$\begin{aligned}\|f\|^2 &= \int_X |f(x)|^2 \mu(dx) < \infty \\ \langle f_1 | f_2 \rangle &= \int_X \overline{f_1(x)} f_2(x) \mu(dx)\end{aligned}$$

Let us select, among elements of $L^2(X, \mu)$, an orthonormal set $\mathcal{S}_N = \{\phi_n(x)\}_{n=1}^N$, N being finite or infinite, which spans, by definition, the separable Hilbert subspace \mathcal{H}_N . We demand this set to obey the following crucial condition

$$0 < \mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.} \quad (14)$$

Then consider the family of states $\{|x\rangle\}_{x \in X}$ in \mathcal{H}_N through the following linear superpositions:

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |\phi_n\rangle. \quad (15)$$

This defines an injective map (which should be continuous w.r.t some minimal topology affected to X for which the latter is locally compact):

$$X \ni x \mapsto |x\rangle \in \mathcal{H}_N,$$

These *coherent* states obey

Normalisation

$$\langle x | x \rangle = 1, \quad (16)$$

Resolution of the unity in \mathcal{H}_N

$$\int_X |x\rangle\langle x| \mathcal{N}(x) \mu(dx) = \mathbb{I}_{\mathcal{H}_N}, \quad (17)$$

A *classical* observable is a function $f(x)$ on X having specific properties. Its coherent state or frame quantization consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x) |x\rangle\langle x| \mathcal{N}(x) \mu(dx). \quad (18)$$

The function $f(x) \equiv \hat{A}_f(x)$ is called upper (or contravariant) symbol of the operator A_f and is nonunique in general. On the other hand, the mean value $\langle x|A_f|x\rangle \equiv \check{A}_f(x)$ is called lower (or covariant) symbol of A_f .

Such a quantization of the set X is in one-to-one correspondence with the choice of the frame

$$\int_X |x\rangle\langle x| \mathcal{N}(x) \mu(dx) = \mathbb{I}_{\mathcal{H}_N}.$$

To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with X (compare with Fourier or wavelet analysis in signal processing). Here, the validity of a precise frame choice is asserted by comparing spectral characteristics of quantum observables A_f with data provided by specific protocols in the observation of X .

Let us now take as a set X the unit circle S^1 provided with the measure $\mu(d\theta) = \frac{d\theta}{2\pi}$. The Hilbert space is $L^2(X, \mu) = L^2(S^1, \frac{d\theta}{2\pi})$ and has the inner product:

$$\langle f|g\rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta) \frac{d\theta}{2\pi}. \quad (19)$$

In this space we choose as orthonormal set the first N Fourier exponentials with negative frequencies:

$$\phi_n(\theta) = e^{-in\theta}, \text{ with } \mathcal{N}(\theta) = \sum_{n=0}^{N-1} |\phi_n(\theta)|^2 = N. \quad (20)$$

The phase states are now defined as the corresponding “coherent states”:

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\theta} |\phi_n\rangle, \quad (21)$$

where the kets $|\phi_n\rangle$ can be directly identified to the number states $|n\rangle$, and the round bracket denotes the continuous labelling of this family. We trivially have normalization and resolution of the unity in $\mathcal{H}_N \simeq \mathbb{C}^N$:

$$(\theta|\theta) = 1, \quad \int_0^{2\pi} |\theta\rangle\langle\theta| N \mu(d\theta) = I_N. \quad (22)$$

Unlike (10) the states (21) are not orthogonal but overlap as:

$$\langle \theta' | \theta \rangle = \frac{e^{i\frac{N-1}{2}(\theta-\theta')}}{N} \frac{\sin \frac{N}{2}(\theta - \theta')}{\sin \frac{1}{2}(\theta - \theta')}. \quad (23)$$

Note that for N large enough these states contain all the Pegg-Barnett phase states and besides they form a continuous family labelled by the points of the circle. The coherent state quantization of a particular function $f(\theta)$ with respect to the continuous set (21) yields the operator A_f defined by:

$$f(\theta) \mapsto \int_X f(\theta) |\theta\rangle \langle \theta| N \mu(d\theta) \stackrel{\text{def}}{=} A_f. \quad (24)$$

An analog procedure has been already used in the frame of positive operator valued measures [8][9] but spanning the phase states over an infinite orthogonal basis with the known drawback on the convergence of the $|\phi\rangle = \sum_n e^{in\theta} |n\rangle$ series out of the Hilbert space and the related questions concerning the operator domain. When expressed in terms of the number states the operator (24) takes the form:

$$A_f = \sum_{n,n'=0}^{N-1} c_{n'-n}(f) |n\rangle \langle n'|, \quad (25)$$

where $c_n(f)$ are the Fourier coefficients of the function $f(\theta)$,

$$c_n(f) = \int_0^{2\pi} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}. \quad (26)$$

Therefore, the existence of the quantum version of f is ruled by the existence of its Fourier transform. Note that A_f will be self-adjoint only when $f(\theta)$ is real valued. In particular, a self-adjoint phase operator of the Toeplitz matrix type, is obtained straightforward by choosing $f(\theta) = \theta$:

$$\hat{A}_\theta = -i \sum_{\substack{n \neq n' \\ n,n'=0}}^{N-1} \frac{1}{n - n'} |n\rangle \langle n'|, \quad (27)$$

Its lower symbol or expectation value in a coherent state is given by:

$$\langle \theta | \hat{A}_\theta | \theta \rangle = \frac{i}{N} \sum_{\substack{n \neq n' \\ n,n'=0}}^{N-1} \frac{e^{i(n-n')\theta}}{n' - n}. \quad (28)$$

Due to the continuous nature of the set of $|\theta\rangle$, all operators produced by this quantization are different of the Pegg-Barnett operators. As a matter of fact, the commutator $[\hat{N}, \hat{A}_\theta]$ expressed in terms of the number basis reads as:

$$[\hat{N}, \hat{A}_\theta] = -i \sum_{\substack{n \neq n' \\ n,n'=0}}^{N-1} |n\rangle \langle n'| = iI_d + (-i)\mathcal{I}_N, \quad (29)$$

and has all diagonal elements equal to 0. Here $\mathcal{I}_N = \sum_{n,n'=0}^{N-1} |n\rangle\langle n'|$ is the $N \times N$ matrix with all entries = 1. The spectrum of this matrix is 0 (degenerate $N-1$ times) and N . The normalized eigenvector corresponding to the eigenvalue N is:

$$|v_N\rangle = |\theta = 0\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle \quad (30)$$

Other eigenvectors span the hyperplane orthogonal to $|v_N\rangle$. We can choose them as the orthonormal set with $N-1$ elements:

$$\left\{ |v_n\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|n+1\rangle - |n\rangle), \quad n = 0, 1, \dots, N-2 \right\}. \quad (31)$$

The matrix \mathcal{I}_N is just N times the projector $|v_N\rangle\langle v_N|$. Hence the commutation rule reads as:

$$[\hat{N}, \hat{A}_\theta] = -i \sum_{\substack{n \neq n' \\ n, n'=0}}^{N-1} |n\rangle\langle n'| = i(I_d - N|v_N\rangle\langle v_N|). \quad (32)$$

A further analysis of this relation through its lower symbol provides, for the matrix \mathcal{I}_N , the function:

$$(\theta|\mathcal{I}_N|\theta) = \frac{1}{N} \sum_{n,n'=0}^{N-1} e^{i(n-n')\theta} = \frac{1}{N} \frac{\sin^2 N\frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}. \quad (33)$$

In the limit at large N this function is the Dirac comb (a well-known result in diffraction theory):

$$\lim_{N \rightarrow \infty} \frac{1}{N} \frac{\sin^2 N\frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} = \sum_{k \in \mathbb{Z}} \delta(\theta - 2k\pi). \quad (34)$$

Recombining this with expression (32) allows to recover the canonical commutation rule:

$$(\theta|[\hat{N}, \hat{A}_\theta]|\theta) \approx_{N \rightarrow \infty} i - i \sum_{k \in \mathbb{Z}} \delta(\theta - 2k\pi). \quad (35)$$

This expression is the expected one for any periodical variable as was seen in (5). It means that in the Heisenberg picture for temporal evolution

$$\hbar \frac{d}{dt} \langle \hat{A}_\theta \rangle = -i \langle [\hat{N}, \hat{A}_\theta] \rangle = 1 - \sum_{k \in \mathbb{Z}} \delta(\theta - 2k\pi) \quad (36)$$

A Dirac commutator-Poisson bracket correspondence can be established from here. The Poisson bracket equation of motion for the phase of the harmonic oscillator is:

$$\frac{d\theta}{dt} = \{H, \theta\} = \omega(1 - \delta(\theta - 2k\pi)), \quad (37)$$

where $H = \frac{1}{2}(p^2 + \omega^2 x^2)$ is the Hamiltonian and $\theta = \arctan(p/\omega x)$ is the phase. The identification $[\hat{N}, \hat{A}_\theta] = i\hbar\omega\{H, \theta\}$ is straightforward and we recover a sawtooth profile for the phase variable just as happened in (5) for the angle variable.

Note that relation (35) is found through the expected value over phase coherent states and not in any physical state as in [7]. This shows that states (21), as canonical coherent states, hold the closest to classical behavior. Another main feature is that any of these states is equal-weighted over the number basis which confirms a total indeterminacy on the eigenstates of the number operator and the opposite is also true. A number state is equal weighted over all the family (21) and in particular this coincides with results in [7].

The creation and annihilation operators are obtained using first the quantization (24) with $f(\theta) = e^{\pm i\theta}$:

$$\hat{A}_{e^{\pm i\theta}} = \int_0^{2\pi} e^{\pm i\theta} N|\theta\rangle\langle\theta| \frac{d\theta}{2\pi}, \quad (38)$$

and then including the number operator as $\hat{A}_{e^{i\theta}} \hat{N}^{\frac{1}{2}} \equiv \hat{a}$ in a similar way to [7] where the authors used instead $e^{i\hat{\theta}_{PB}} \hat{N}^{\frac{1}{2}}$. The commutation relation between both operators is

$$[\hat{a}, \hat{a}^\dagger] = 1 - N|N-1\rangle\langle N-1|, \quad (39)$$

which converges to the common result only when the expectation value is taken on states where extremal state component vanish as n tends to infinity.

As the phase operator is not built from a spectral decomposition, it is clear that $\hat{A}_{\theta^2} \neq \hat{A}_\theta^2$ and the link with an uncertainty relation is not straightforward as in [7], instead, as is suggested in [9], a different definition for the variance should be used.

The phase operator constructed here has most of the advantages of the Pegg-Barnett operator but allows more freedom within the Hilbertian framework. It is clear that a well-defined phase operator must be parametrised by all points in the circle in order to have a natural convergence to the commutation relation in the classical limit. It remains also clear that as in any measure, like Pegg-Barnett's or this one through coherent states, the inconveniences due to the non periodicity of the phase pointed in [3] are avoided from the very beginning in the choice of $X \equiv S^1$.

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